

# A SPECTRAL THEORY OF NON-UNIFORMLY CONTINUOUS FUNCTIONS AND THE LOOMIS-ARENDT-BATTY-VU THEORY ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF EVOLUTION EQUATIONS

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ABSTRACT. In this paper we present a new approach to the spectral theory of *non-uniformly continuous* functions and a new framework for the Loomis-Arendt-Batty-Vu theory. Our approach is direct and free of  $C_0$ -semigroups, so the obtained results, that extend previous ones, can be applied to large classes of evolution equations and their solutions.

## 1. INTRODUCTION

There is a remarkable theory of the asymptotic behavior of orbits of bounded one-parameter semigroups of operators on Banach spaces in the case when the purely imaginary part of the spectrum of the generator is countable. Although it can be viewed as part of Pure Functional Analysis and Operator Theory, it has direct applications to the asymptotic behavior of solutions of evolution equations. This theory is closely related to Tauberian Theory, Loomis' theorem in Harmonic Analysis concerning almost periodicity of functions with countable Beurling spectrum, and a Gelfand's theorem on the spectrum of  $C_0$ -groups of isometries.

Significant contributions in the theory which we refer to as the Loomis-Arendt-Batty-Vu theory, were made by L. Loomis, W. Arendt, C. Batty, Vu Quoc Phong and others. We refer the reader to the monographs [4, 39] and their references for more systematic information on the theory. The Loomis-Arendt-Batty-Vu theory is based on early works extending Loomis' theorem to study almost periodic solutions of evolution equations, [25, 10, 11]. In this direction, general results of the type of Loomis' theorem were proved by applying a Gelfand's theorem, and the concept of reduced Beurling spectrum (see e.g. [3, 6, 9, 10, 11, 25, 45]). A far-reaching ergodic condition, introduced in [45] to study asymptotically almost periodic functions on the line (see also [6]) turns out to be a key tool to present the theory in a unified framework, especially for asymptotic almost periodicity, asymptotic stability of solutions of evolution equations on the half line (see e.g. [2, 3, 4, 13, 15, 18, 29, 30, 41]).

There is a "prevalence of the hypothesis of uniform continuity in the literature", as remarked by Loomis in the Introduction of his paper [28]. This remark seems to have been valid until now, and can be explained by the use of sophisticated tools involving semigroups of operators. As the Loomis-Arendt-Batty-Vu theory has direct applications to evolution equations, the requirement

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that the equations be well-posed or the solutions be uniformly continuous for the use of semigroup theory seems to be technical, and is an obstacle for applications.

In this paper we will take an attempt to push forward the Loomis-Arendt-Batty-Vu theory by introducing a new approach to the spectral theory of *not necessarily uniformly continuous functions*, and a new framework for the main points of the theory. Our approach is more elementary, free of  $C_0$ -semigroups. As a result, the theory with general results can be presented in a way that is easily accessible to readers working in ordinary differential equations with a limited knowledge of the methods of harmonic analysis.

We will start the paper with a new approach to the concept of reduced spectrum of a bounded function on  $\mathbb{R}$  or on  $\mathbb{R}^+$  that is *not necessarily uniformly continuous*. This approach is based on some simple facts from the ODE that gives more direct insights into the differentiation operator  $\mathcal{D} := d/dt$  on various function spaces. As a result, a new relation between the reduced spectrum and spectrum of the differentiation operator  $\mathcal{D}$  is established (Theorem 2.12). We also extend the Gelfand's theorem (Theorem 2.18) as the key tool to prove Loomis Theorem (Theorem 2.25). Among consequences of the Loomis Theorem we mention standard ones without the hypothesis of uniform continuity (Corollaries 2.27, 2.29). As an almost automorphic function may not be uniformly continuous, Corollary 2.31 is a typical example of a result which has not been covered in previous works. We emphasize that a version of Loomis Theorem with an ergodic condition can be easily derived following the lines discussed in the next part of the paper. For functions on the half line, our direct approach to the differentiation operator  $\mathcal{D}$  on  $\mathbb{R}^+$  results in Theorem 3.5 which is a key tool to prove the main result of the section (Theorem 3.6). In the applications of our results to stability of evolution equations we can free both the hypotheses of uniform continuity and well-posedness (Corollary 4.3, Theorem 4.11). In particular, our approach also yields the well known Arendt-Batty-Lyubich-Vu Theorem (Corollary 4.14). In our references we give a (non-exhausted) list of works concerned with the theory and related applications of spectral theory of functions to the asymptotic behavior of evolution equations. (The reader may use the references of [4, 39, 42] for a more complete list of references for this paper.)

The first version of this paper was announced in the preprint [32] (see also [33, 27] for a related result). In this new version, we have added some new developments from the literature since then with appropriate comments as well as corrections and applications.<sup>1</sup>

Before closing this section we would like to list some standard notations we will use in the paper. Throughout the paper  $\mathbb{R}$  denotes the real line,  $\mathbb{R}^+$  denotes the half line  $[0, \infty)$ , and  $\mathbb{X}$  denotes a Banach space over the complex plane  $\mathbb{C}$ . If  $A$  is a linear operator on a Banach space  $\mathbb{X}$ ,  $D(A)$  stands for its domain;  $\sigma(A)$ ,  $\sigma_p(A)$ ,  $\rho(A)$  stand for its spectrum, point spectrum and resolvent set, respectively.  $L(\mathbb{X})$  stands for the Banach space of all bounded linear operators in  $\mathbb{X}$  with the usual

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<sup>1</sup>After the first version [32] of this paper appeared we learned that in [8] related concepts are discussed. However, the related results stated in [8] seem to be different and incomplete. For example, in addition to the uniform closedness condition which seems to be very strict, Proposition 4.1 and Theorem 4.3 obviously assume that the reduced spectrum of the considered function is non-empty. This condition appears to be significant if the Gelfand Theorem for isometries is not available.

norm  $\|\cdot\|$ . If  $\lambda \in \rho(A)$ , then  $R(\lambda, A)$  denotes the resolvent  $(\lambda - A)^{-1}$ . In this paper we will use the following notations:

- i)  $c_0$  denotes the Banach space of all numerical sequence  $x = \{x_n\}_{n=1}^\infty$  such that  $\lim_{n \rightarrow \infty} x_n = 0$ , with sup-norm  $\|x\| = \sup_{n \in \mathbb{N}} \|x_n\|$ ;
- ii)  $J$  is either  $\mathbb{R}$  or  $\mathbb{R}^+$ ;
- iii)  $BC(J, \mathbb{X})$  is the space of all  $\mathbb{X}$ -valued bounded and continuous functions on  $J$ ;
- iv)  $BUC(J, \mathbb{X})$  is the space of all  $\mathbb{X}$ -valued bounded and uniformly continuous functions on  $J$ ;
- v)  $AP(\mathbb{X})$ ,  $AP(\mathbb{R}^+, \mathbb{X})$  are the spaces of all  $\mathbb{X}$ -valued almost periodic functions on  $\mathbb{R}$ , and  $\mathbb{R}^+$ , respectively;
- vi)  $AA(\mathbb{X})$ ,  $AA(\mathbb{R}^+, \mathbb{X})$  are the spaces of all  $\mathbb{X}$ -valued almost automorphic functions on  $\mathbb{R}$ , and  $\mathbb{R}^+$ , respectively;
- vii)  $C_0(J, \mathbb{X}) := \{f \in BC(J, \mathbb{X}) : \lim_{t \rightarrow \infty} f(t) = 0\}$ ;
- viii)  $AAP(\mathbb{R}^+, \mathbb{X}) := C_0(\mathbb{R}^+, \mathbb{X}) \oplus AP(\mathbb{X})$ ;
- ix) If  $A$  is a linear operator on  $\mathbb{X}$ , then the operator of multiplication by  $A$  on  $BC(J, \mathbb{X})$ , denoted by  $\mathcal{A}$ , is defined on  $D(\mathcal{A}) := \{g \in BC(J, \mathbb{X}) : g(t) \in D(A), \text{ for all } t \in J, Ag(\cdot) \in BC(J, \mathbb{X})\}$ , by  $\mathcal{A}g = Ag(\cdot)$  for each  $g \in D(\mathcal{A})$ .

In this paper, by almost periodic functions we mean the ones in the sense of Bohr (for the precise definition and properties see e.g. [25]), and by almost automorphic functions we mean the ones in the sense of Bochner (for the precise definition and properties see e.g. [16, 40, 46, 49, 50]).

## 2. A SPECTRAL THEORY OF BOUNDED FUNCTIONS ON THE LINE

**2.1. Reduced spectrum.** Let us introduce some operators and discuss the relations between their resolvent sets and spectra of a bounded function.

**Definition 2.1.**  $D(\mathcal{D})$  is defined to be the set of all differentiable functions  $f \in BC(\mathbb{R}, \mathbb{X})$  such that  $f' \in BC(\mathbb{R}, \mathbb{X})$ . The operator  $\mathcal{D}$  is defined by  $\mathcal{D}f = f'$  whenever  $f \in D(\mathcal{D})$ .

Some elementary properties of the operator  $\mathcal{D}$  are summarized in the following lemma.

**Lemma 2.2.**  $\mathcal{D}$  is a closed operator on  $BC(\mathbb{R}, \mathbb{X})$  with  $\sigma(\mathcal{D}) = i\mathbb{R}$ . Moreover, for each  $\xi \in \mathbb{R}$ ,  $Re\lambda \neq 0$ ,

$$(2.1) \quad R(\lambda, \mathcal{D})f(\xi) = \begin{cases} \int_0^\infty e^{-\lambda\eta} f(\xi + \eta) d\eta & (\text{if } Re\lambda > 0) \\ -\int_{-\infty}^0 e^{-\lambda\eta} f(\xi + \eta) d\eta & (\text{if } Re\lambda < 0). \end{cases}$$

*Proof.* Since for each  $\lambda \in \mathbb{C}$  such that  $Re\lambda \neq 0$ , the differential equation

$$(2.2) \quad x'(t) - \lambda x(t) = 0, \quad x(t) \in \mathbb{C}$$

has an exponential dichotomy, the non-homogeneous equation

$$(2.3) \quad x'(t) - \lambda x(t) = f(t), \quad x(t) \in \mathbb{C}$$

has a unique solution  $x_{f,\lambda} \in BC(\mathbb{R}, \mathbb{X})$  for each given  $f \in BC(\mathbb{R}, \mathbb{X})$ . It is well known in the theory of ODE that  $x_{f,\lambda}$  is determined by the Green operator, that is,

$$(2.4) \quad \begin{aligned} -x_{f,\lambda}(\xi) &= \begin{cases} \int_{\xi}^{\infty} e^{\lambda(\xi-t)} f(t) dt & (\text{if } \operatorname{Re} \lambda > 0) \\ -\int_{-\infty}^{\xi} e^{\lambda(\xi-t)} f(t) dt & (\text{if } \operatorname{Re} \lambda < 0). \end{cases} \\ &= \begin{cases} \int_0^{\infty} e^{-\lambda\eta} f(\xi + \eta) d\eta & (\text{if } \operatorname{Re} \lambda > 0) \\ -\int_{-\infty}^0 e^{-\lambda\eta} f(\xi + \eta) d\eta & (\text{if } \operatorname{Re} \lambda < 0). \end{cases} \end{aligned}$$

Therefore,  $-x_{f,\lambda}$  is a bounded linear operator in  $f$  acting in  $BC(\mathbb{R}, \mathbb{X})$ , that is,  $R(\lambda, \mathcal{D})$  exists and  $R(\lambda, \mathcal{D})f = -x_{f,\lambda}$ .

Next, note that for each  $i\xi$ , where  $\xi \in \mathbb{R}$ , the function  $f_{\xi}(t) := e^{i\xi t}x$ , where  $x \in \mathbb{X}$  is a non-zero element, is an eigenvector of  $\mathcal{D}$ . Thus,  $\sigma(\mathcal{D}) = i\mathbb{R}$ . As a consequence,  $\rho(\mathcal{D}) \neq \emptyset$ , so  $\mathcal{D}$  is closed. Formula (2.1) follows from (2.4).  $\square$

Let  $\mathcal{F}$  be a closed subspace of  $BC(\mathbb{R}, \mathbb{X})$  that satisfies the following condition:

**Definition 2.3.** *A closed subspace  $\mathcal{F}$  of  $BC(\mathbb{R}, \mathbb{X})$  is said to satisfy Condition F if either it is trivial, or it satisfies the following condition:*

- i) *It contains all constant functions;*
- ii) *If  $f \in \mathcal{F}$ , then, for each  $\xi \in \mathbb{R}$ , the function  $f_{\xi}$  defined as  $f_{\xi}(t) = e^{i\xi t}f(t)$ , for all  $t \in \mathbb{R}$ , belongs to  $\mathcal{F}$ ;*
- iii) *For each  $\lambda \in \mathbb{C}$  such that  $\operatorname{Re} \lambda \neq 0$ , one has*

$$(2.5) \quad R(\lambda, \mathcal{D})\mathcal{F} \subset \mathcal{F};$$

- iv) *If  $f \in \mathcal{F}$  and  $F$  is a bounded primitive of  $f$ , then  $F \in \mathcal{F}$ ;*
- v) *For each  $B \in L(\mathbb{X})$  and  $f \in \mathcal{F}$ , the function  $Bf(\cdot)$  is in  $\mathcal{F}$ .*

**Remark 2.4.** i) *In the case when  $\mathcal{F}$  is a subspace of  $BUC(\mathbb{R}, \mathbb{X})$ , condition (iii) follows from the translation invariance of  $\mathcal{F}$ . In fact, this follows from the representation*

$$R(\lambda, \mathcal{D})g = \int_0^{\infty} e^{-\lambda t} S(t)g dt, \quad \operatorname{Re} \lambda > 0, t \in \mathbb{R},$$

or

$$R(\lambda, \mathcal{D})g = -\int_0^{\infty} e^{\lambda t} S(-t)g dt, \quad \operatorname{Re} \lambda < 0, t \in \mathbb{R}.$$

- ii) *Condition iv) in the case  $\mathcal{F} = AP(\mathbb{X})$  is the validity of the well known Bohl-Bohr-Kadets Theorem (see e.g. [4, 25]). For example, condition (iv) holds in this case if  $\mathbb{X}$  does not contain any subspace isomorphic to  $c_0$ . For related concepts see [7, Definition 3.1]<sup>2</sup>*

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<sup>2</sup>Compared with the conditions listed in [8, (3.1)] to define the concept of reduced spectrum our conditions are different. The uniform-closedness of  $\mathcal{A}$  as a condition of [7, Definition 3.1] seems to be very strict. In fact, it can be easily checked that such a space  $\mathcal{A}$  cannot be  $BUC(\mathbb{R}, \mathbb{X})$ ,  $BC(\mathbb{R}, \mathbb{X})$

**Example 2.5.** Another example of such a function space  $\mathcal{F}$  is the space of all almost automorphic functions  $AA(\mathbb{X})$ . It is known (see e.g. [34]) that  $AA(\mathbb{X}) \not\subset BUC(\mathbb{R}, \mathbb{X})$ . Therefore, the condition iii) does not follow from the translation invariance of  $AA(\mathbb{X})$ . This condition can be checked directly. A particular case of the main result in [34, Theorem 3.2] yields that the condition iv) is fulfilled for this function space if  $\mathbb{X}$  do not contain any subspace isomorphic to  $c_0$ .

Consider the quotient space  $\mathbb{Y} := BC(\mathbb{R}, \mathbb{X})/\mathcal{F}$ , where  $\mathcal{F}$  is a given closed subspace of  $BC(\mathbb{R}, \mathbb{X})$  that satisfies Condition F. Every element of this quotient space is a class of functions in  $BC(\mathbb{R}, \mathbb{X})$  that is denoted by  $\tilde{f}$ , where  $f \in BC(\mathbb{R}, \mathbb{X})$ .

**Definition 2.6.**  $D(\tilde{\mathcal{D}})$  is defined to be the subset of  $\mathbb{Y}$  consisting of all classes of functions that contain elements of  $D(\mathcal{D})$ . The operator  $\tilde{\mathcal{D}}$  is defined by  $\tilde{\mathcal{D}}\tilde{f} = \tilde{f}'$  whenever  $\tilde{f} \in D(\tilde{\mathcal{D}})$  containing  $f \in D(\mathcal{D})$ .

Some elementary properties of  $\tilde{\mathcal{D}}$  are summarized in the following

**Lemma 2.7.**  $\tilde{\mathcal{D}}$  is a closed linear operator on  $\mathbb{Y}$  with  $\sigma(\tilde{\mathcal{D}}) \subset \sigma(\mathcal{D}) = i\mathbb{R}$ . Moreover, for  $Re\lambda \neq 0$ ,

$$(2.6) \quad \|R(\lambda, \tilde{\mathcal{D}})\| \leq \|R(\lambda, \mathcal{D})\| \leq \frac{1}{|Re\lambda|}.$$

*Proof.* Take any  $\lambda \in \rho(\mathcal{D})$ , that is,  $Re\lambda \neq 0$ . We will show that  $\lambda \in \rho(\tilde{\mathcal{D}})$ . In fact, for any  $f \in BC(\mathbb{R}, \mathbb{X})$ , since  $R(\lambda, \mathcal{D})\mathcal{F} \subset \mathcal{F}$ , we have  $R(\lambda, \mathcal{D})\tilde{f}$  is contained in  $\tilde{g}$  defined as the class containing  $R(\lambda, \mathcal{D})f$ . So, the equation

$$(2.7) \quad \lambda\tilde{g} - \tilde{\mathcal{D}}\tilde{g} = \tilde{f}$$

has at least one solution as the class containing  $R(\lambda, \mathcal{D})f$  for each given  $\tilde{f} \in \mathbb{Y}$ . Moreover, we have

$$\begin{aligned} \|\tilde{g}\|_{\mathbb{Y}} &:= \inf_{h \in \mathcal{F}} \|R(\lambda, \mathcal{D})f + h\| \\ &\leq \inf_{k \in \mathcal{F}} \|R(\lambda, \mathcal{D})(f + k)\| \\ &\leq \|R(\lambda, \mathcal{D})\| \inf_{k \in \mathcal{F}} \|f + k\| \\ &= \|R(\lambda, \mathcal{D})\| \cdot \|f\|_{\mathbb{Y}}. \end{aligned}$$

Now we show that (2.7) has no more than one solution. Indeed, it is equivalent to show that the homogeneous equation has no solution other than the zero solution. In fact, if  $\lambda\tilde{g} - \tilde{\mathcal{D}}\tilde{g} = 0$ , then, assuming  $\tilde{g}$  contains  $g \in D(\mathcal{D})$ , we have

$$\lambda g(t) - g'(t) = h(t), \quad \text{for all } t \in \mathbb{R},$$

where  $h$  is a function in  $\mathcal{F}$ . However, since  $h \in \mathcal{F}$  from the condition iii) of Condition F,  $g$  must be in  $\mathcal{F}$ , so  $\tilde{g} = 0$ .

Summing up all we have done above shows that  $\rho(\tilde{\mathcal{D}}) \supset \rho(\mathcal{D}) = \mathbb{C} \setminus i\mathbb{R} \neq \emptyset$ . As a consequence,  $\tilde{\mathcal{D}}$  is closed. The estimate (2.6) follows from the above estimate of  $\|R(\lambda, \tilde{\mathcal{D}})\|$  in terms of  $\|R(\lambda, \mathcal{D})\|$ , and in turn, an estimate of  $\|R(\lambda, \mathcal{D})\|$  from (2.1)  $\square$

We are ready to define the concept of reduced spectrum of a bounded function.

**Definition 2.8.** Let  $\mathcal{F}$  be a closed subspace function of  $BC(\mathbb{R}, \mathbb{X})$  that satisfies Condition F, and let  $f \in BC(\mathbb{R}, \mathbb{X})$ . Then, the reduced spectrum of  $f$  with respect to  $\mathcal{F}$ , denoted by  $sp_{\mathcal{F}}(f)$ , is defined to be the set of all reals  $\xi$  such that the complex function  $R(\lambda, \tilde{\mathcal{D}})f$ , as a function of  $\lambda \in \mathbb{C} \setminus i\mathbb{R}$ , has no analytic extension to any neighborhood of  $i\xi$  in  $\mathbb{C}$ . If  $\mathcal{F}$  is trivial, we use the notation  $sp(f)$  instead of  $sp_0(f)$ , and call it simply spectrum of  $f$ .

**Remark 2.9.** If  $f \in BUC(\mathbb{R}, \mathbb{X})$ , and  $\mathcal{F}$  is a subspace of  $BUC(\mathbb{R}, \mathbb{X})$  that satisfies Condition F, then  $sp_{\mathcal{F}}(f)$  is the reduced spectrum of  $f$  as defined in [2, 4, 6, 10, 45].

**Proposition 2.10.** Let  $\mathcal{F}$  be a closed subspace of  $BC(\mathbb{R}, \mathbb{X})$  that satisfies Condition F. Then the following assertions hold:

- i)  $sp_{\mathcal{F}}(f)$  is closed, and  $sp_{\mathcal{F}}(f) = sp_{\mathcal{F}}(g)$  for all  $g \in \tilde{f}$ ;
- ii)  $sp_{\mathcal{F}}(f) = sp_{\mathcal{F}}(f(\cdot + c))$ , for any  $c \in \mathbb{R}$ ,  $f \in BC(\mathbb{R}, \mathbb{X})$ ;
- iii)  $sp_{\mathcal{F}}(Af(\cdot)) \subset sp_{\mathcal{F}}(f)$  for each  $f \in BC(\mathbb{R}, \mathbb{X})$ ,  $A \in L(\mathbb{X})$ ;
- iv)  $sp_{\mathcal{F}}(f + g) \subset sp_{\mathcal{F}}(f) \cup sp_{\mathcal{F}}(g)$  for all  $f, g \in BC(\mathbb{R}, \mathbb{X})$ ;
- v)  $sp_{\mathcal{F}}(f) \subset \Lambda$  if there are  $f_n \in BC(\mathbb{R}, \mathbb{X})$ ,  $n \in \mathbb{N}$ , such that  $\tilde{f}_n \rightarrow \tilde{f} \in \mathbb{Y} := BC(\mathbb{R}, \mathbb{X})/\mathcal{F}$ ,  $sp_{\mathcal{F}}(f_n) \subset \Lambda$  for all  $n$ , where  $\Lambda$  is a closed subset of  $\mathbb{R}$ ;
- vi)  $sp_{\mathcal{F}}(f) \subset sp(f)$ .

*Proof.* Properties i-iv) follows immediately from the definition. We now prove v). Let  $\rho_0 \notin \Lambda$ . Since  $\Lambda$  is closed, there is a positive constant  $r < \text{dist}(\rho_0, \Lambda)$ . As in the proof of [43, Theorem 0.8, p. 21] or by [4, Lemma 4.6.6, p. 295] we can prove that since  $R(\lambda, \tilde{\mathcal{D}})\tilde{f}_n$  is extendable to all  $\bar{B}_r(i\rho_0)$ , and

$$(2.8) \quad \|R(\lambda, \tilde{\mathcal{D}})\tilde{f}_n\| \leq \frac{2\|\tilde{f}\|}{|Re\lambda|}, \quad \text{for all } \lambda \in \bar{B}_r(i\rho_0)$$

for sufficiently large  $n \geq N$ , one has

$$(2.9) \quad \|R(\lambda, \tilde{\mathcal{D}})\tilde{f}_n\| \leq \frac{4\|\tilde{f}\|}{3r}, \quad \text{for all } \lambda \in \bar{B}_r(i\rho_0), n \geq N.$$

Obviously, for every fixed  $\lambda$  such that  $Re\lambda \neq 0$  we have  $R(\lambda, \tilde{\mathcal{D}})\tilde{f}_n \rightarrow R(\lambda, \tilde{\mathcal{D}})\tilde{f}$ . Now applying Vitali's theorem to the sequence of complex functions  $\{R(\lambda, \tilde{\mathcal{D}})\tilde{f}_n\}$  we see that  $R(\lambda, \tilde{\mathcal{D}})\tilde{f}_n$  is convergent uniformly on  $B_r(i\rho_0)$  to  $R(\lambda, \tilde{\mathcal{D}})\tilde{f}$ . This yields that  $R(\lambda, \tilde{\mathcal{D}})\tilde{f}$  is holomorphic on  $B_r(i\rho_0)$ , that is,  $\rho_0 \notin sp_{\mathcal{F}}(f)$ .

For vi) it is obvious since the canonical projection on the quotient space  $BC(\mathbb{R}, \mathbb{X}) \rightarrow BC(\mathbb{R}, \mathbb{X})/\mathcal{F}$  is continuous and  $p(R(\lambda, \mathcal{D})f) = R(\lambda, \tilde{\mathcal{D}})\tilde{f}$  for each  $f \in BC(\mathbb{R}, \mathbb{X})$  and  $Re\lambda \neq 0$ .  $\square$

**Corollary 2.11.** Let  $\mathcal{F}$  be a closed subspace of  $BC(\mathbb{R}, \mathbb{X})$  that satisfies Condition F, and let  $\Lambda$  be a closed subset of  $\mathbb{R}$ . Then the function space

$$(2.10) \quad \Lambda_{\mathcal{F}}(\mathbb{X}) := \{\tilde{f} \in \mathbb{Y} := BC(\mathbb{R}, \mathbb{X})/\mathcal{F} : sp_{\mathcal{F}}(\tilde{f}) \subset \Lambda\}$$

is a closed subspace of  $\mathbb{Y}$  that satisfies

$$R(\eta, \tilde{\mathcal{D}})\Lambda_{\mathcal{F}}(\mathbb{X}) \subset \Lambda_{\mathcal{F}}(\mathbb{X}), \quad \text{for all } Re\eta \neq 0.$$

*Proof.* The first assertion of the corollary follows from Properties i-v) of Proposition 2.10. The last one follows from the note that for  $Re\lambda \neq 0$  and  $Re\eta \neq 0$ ,

$$R(\lambda, \tilde{\mathcal{D}})R(\eta, \tilde{\mathcal{D}})\tilde{f} = R(\eta, \tilde{\mathcal{D}})R(\lambda, \tilde{\mathcal{D}})\tilde{f}.$$

□

Let  $\mathcal{F}$  be a closed subspace of  $BC(\mathbb{R}, \mathbb{X})$  that satisfies Condition F, and let  $\Lambda$  be a closed subset of  $\mathbb{R}$ . Then, we define an operator  $\tilde{\mathcal{D}}_\Lambda$  on  $\Lambda_{\mathcal{F}}(\mathbb{X})$  that is the part of  $\tilde{\mathcal{D}}$  on  $\Lambda_{\mathcal{F}}(\mathbb{X})$ , that is,

$$D(\tilde{\mathcal{D}}_\Lambda) := \{\tilde{f} \in D(\tilde{\mathcal{D}}) \cap \Lambda_{\mathcal{F}}(\mathbb{X}) : \tilde{\mathcal{D}}\tilde{f} \in \Lambda_{\mathcal{F}}(\mathbb{X})\}$$

and  $\tilde{\mathcal{D}}_\Lambda\tilde{f} = \tilde{\mathcal{D}}\tilde{f}$ , whenever  $\tilde{f} \in D(\tilde{\mathcal{D}}_\Lambda)$ . If  $\mathcal{F}$  is trivial, we will use the notation  $\mathcal{D}_\Lambda$  instead of  $\tilde{\mathcal{D}}_\Lambda$ .

**Theorem 2.12.** *Let  $\mathcal{F}$  be a closed nontrivial subspace of  $BC(\mathbb{R}, \mathbb{X})$  that satisfies Condition F. Then,*

$$(2.11) \quad \sigma(\tilde{\mathcal{D}}_\Lambda) \subset i\Lambda.$$

*Proof.* This is equivalent to show that every  $\beta \in \mathbb{R} \setminus \Lambda$  is in  $\rho(\mathcal{D}_\Lambda)$ , that is, the following equation

$$(2.12) \quad i\beta\tilde{g} - \tilde{g}' = \tilde{f}$$

is solvable uniquely in  $\Lambda_{\mathcal{F}}(\mathbb{X})$  for every given  $\tilde{f} \in \Lambda_{\mathcal{F}}(\mathbb{X})$ . First, we show that (2.12) has at most one solution, or equivalently, the homogeneous equation  $i\beta\tilde{g} - \tilde{g}' = 0$  has zero as the unique solution in  $\Lambda_{\mathcal{F}}(\mathbb{X})$ . In fact, if  $\tilde{g}$  is a solution of this homogeneous equation, then

$$i\beta g - g' = h \in \mathcal{F}.$$

By the Variation-of-Constants Formula this equation is equivalent to the following

$$g(t) = e^{i\beta t}g(0) - \int_0^t e^{i\beta(t-\xi)}h(\xi)d\xi, \quad \text{for all } t \in \mathbb{R}.$$

Therefore,

$$e^{-i\beta t}g(t) = g(0) - \int_0^t e^{-i\beta\xi}h(\xi)d\xi, \quad \text{for all } t \in \mathbb{R}.$$

Since  $h \in \mathcal{F}$  and  $F$  satisfies Condition F, we see that the function  $\mathbb{R} \ni \xi \mapsto e^{-i\beta\xi}h(\xi)$  belongs to  $\mathcal{F}$ . Moreover, the function  $\mathbb{R} \ni t \mapsto g(0) - e^{-i\beta t}g(t)$  is a bounded primitive of the previous one, so by the item i) of Condition F, the function  $\mathbb{R} \ni t \mapsto e^{-i\beta t}g(t)$  belongs to  $\mathcal{F}$ . Therefore,  $g$  also belongs to  $\mathcal{F}$ , that is,  $\tilde{g} = 0$ .

Next, we show that (2.12) has at least one solution. For every  $Re\lambda \neq 0$  the equation

$$\lambda y - y' = \tilde{f}$$

has a unique solution  $\tilde{g}_\lambda = R(\lambda, \tilde{\mathcal{D}})\tilde{f}$ , that is in  $\Lambda_{\mathcal{F}}(\mathbb{X})$  by Corollary 2.11. Since  $R(\eta, \tilde{\mathcal{D}})\tilde{f}$  is analytic around  $i\beta$ ,  $\lim_{\lambda \rightarrow i\beta} R(\lambda, \tilde{\mathcal{D}})\tilde{f}$  exists as an element, say,  $\tilde{g} \in \Lambda_{\mathcal{F}}(\mathbb{X})$ . Now we show that  $\tilde{g}$  is a solution of (2.12). Indeed, since

$$\begin{aligned} (i\beta - \tilde{\mathcal{D}})R(\lambda, \tilde{\mathcal{D}})\tilde{f} &= ((i\beta - \lambda) + (\lambda - \tilde{\mathcal{D}}))R(\lambda, \tilde{\mathcal{D}})\tilde{f} \\ &= (i\beta - \lambda)R(\lambda, \tilde{\mathcal{D}})\tilde{f} + (\lambda - \tilde{\mathcal{D}})R(\lambda, \tilde{\mathcal{D}})\tilde{f} \\ &= (i\beta - \lambda)R(\lambda, \tilde{\mathcal{D}})\tilde{f} + \tilde{f}, \end{aligned}$$

and  $R(\lambda, \tilde{\mathcal{D}})\tilde{f}$  has an analytic extension around  $i\beta$ , we have

$$(2.13) \quad \lim_{\lambda \rightarrow i\beta} (i\beta - \tilde{\mathcal{D}})R(\lambda, \tilde{\mathcal{D}})\tilde{f} = f.$$

By the closedness of the operator  $(i\beta - \tilde{\mathcal{D}})$ , we come up with  $\tilde{g}$  being in the domain of  $i\beta - \tilde{\mathcal{D}}$  and  $(i\beta - \tilde{\mathcal{D}})\tilde{g} = \tilde{f}$ .  $\square$

**Remark 2.13.** When  $\mathcal{F} = \{0\}$ , it is proved in [26] that  $\sigma(\mathcal{D}_\Lambda) = i\Lambda$ . We refer the reader to [17] for more related results concerned with the case  $f \in BUC(\mathbb{R}, \mathbb{X})$ . Results of this type can be used to study the existence and uniqueness of bounded solutions to non-homogeneous equations (see [35, 36, 19, 26, 38]).

**2.2. Coincidence of the notions of spectrum.** We first recall some concepts.

**Definition 2.14.** The Carleman spectrum of  $f \in BC(\mathbb{R}, \mathbb{X})$  is defined to be the set of all reals  $\xi$  such that the Carleman transform

$$(2.14) \quad \hat{f}(\lambda) := \begin{cases} \int_0^\infty e^{-\lambda\eta} f(\eta) d\eta & (\text{if } \operatorname{Re} \lambda > 0) \\ -\int_{-\infty}^0 e^{-\lambda\eta} f(\eta) d\eta & (\text{if } \operatorname{Re} \lambda < 0), \end{cases}$$

as a complex function of  $\lambda$ , has no analytic extension to any neighborhood of  $i\xi$ .

From the definition of Carleman spectrum of  $f \in BC(\mathbb{R}, \mathbb{X})$ , that will be denoted by  $sp_c(f)$ , it is clear that  $sp_c(f) \subset sp(f)$ .

**Definition 2.15.** Let  $f \in BC(\mathbb{R}, \mathbb{X})$ . The Beurling spectrum of  $f$ , that is denoted by  $sp_b(f)$ , is defined to be the following set

$$(2.15) \quad sp_b(f) := \{\xi \in \mathbb{R} : \text{ for all } \epsilon > 0, \exists \phi \in L^1(\mathbb{R}), \operatorname{supp}(\tilde{\phi}) \subset (\xi - \epsilon, \xi + \epsilon), \phi * f \neq 0\},$$

where

$$\tilde{\phi}(\eta) := \int_{-\infty}^\infty e^{i\eta t} \phi(t) dt, \quad t \in \mathbb{R},$$

is the Fourier transform of  $\phi$ , and

$$\phi * f(t) := \int_{-\infty}^\infty \phi(s) f(t - s) ds, \quad \text{for all } t \in \mathbb{R}.$$

**Proposition 2.16.** Let  $f \in BC(\mathbb{R}, \mathbb{X})$ . Then

$$(2.16) \quad sp(f) = sp_c(f) = sp_b(f).$$

*Proof.* For the proof of  $sp_c(f) = sp_b(f)$  see [4, Proposition 4.8.4, p. 321] and [43]. For the identity  $sp(f) = sp_c(f) = sp_b(f)$  see [26].  $\square$



**2.3. Loomis Theorem.** The main result we will prove in the section is of the Loomis Theorem type for general classes of functions. Before doing so, we need some preparatory results. The following lemma is known.

**Lemma 2.17.** *Let  $f(z)$  be a complex function taking values in a Banach space  $\mathbb{X}$  and be holomorphic in  $\mathbb{C} \setminus i\mathbb{R}$  such that there is a positive number  $M$  independent of  $z$  for which*

$$(2.17) \quad \|f(z)\| \leq \frac{M}{|\operatorname{Re} z|}, \quad \text{for all } \operatorname{Re} z \neq 0.$$

*Assume further that  $i\xi \in i\mathbb{R}$  is an isolated singular point of  $f(z)$  at which the Laurent expansion is of the form*

$$(2.18) \quad f(z) = \sum_{n=-\infty}^{\infty} a_n(z - i\xi)^n,$$

where

$$(2.19) \quad a_n = \frac{1}{2\pi i} \int_{|z-i\xi|=r} \frac{f(z)dz}{(z-i\xi)^{n+1}}, \quad n \in \mathbb{Z}.$$

Then,

$$(2.20) \quad \|r^2 a_{-(n+1)} + a_{-(n+3)}\| \leq 2Mr^{n+2}, \quad n \in \mathbb{Z}.$$

*Proof.* Note that the below proof can be found in [4, Lemma 4.6.6], [43, Chap. 0]. For the reader's convenience we reproduce it here.

For each  $n \in \mathbb{Z}$  and  $0 < r < \delta_0$ , where  $\delta_0$  is some positive number, we have

$$\begin{aligned} & \left\| \frac{1}{2\pi i} \int_{|z-i\xi|=r} (z-i\xi)^n \left( 1 + \frac{(z-i\xi)^2}{r^2} \right) f(z) dz \right\| \\ & \leq \frac{1}{2\pi} \int_{|z-i\xi|=r} \left| (z-i\xi)^n \left( 1 + \frac{(z-i\xi)^2}{r^2} \right) \right| \cdot \|f(z)\| \cdot |dz|. \end{aligned}$$

A simple computation shows that since  $|z-i\xi|=r$ , one has

$$(2.21) \quad \left| (z-i\xi)^n \left( 1 + \frac{(z-i\xi)^2}{r^2} \right) \right| = 2r^{n-1} |\operatorname{Re} z|.$$

Therefore,

$$\begin{aligned} \left\| \frac{1}{2\pi i} \int_{|z-i\xi|=r} (z-i\xi)^n \left( 1 + \frac{(z-i\xi)^2}{r^2} \right) f(z) dz \right\| & \leq \frac{1}{2\pi} \int_{|z-i\xi|=r} 2r^{n-1} |\operatorname{Re} z| \frac{M}{|\operatorname{Re} z|} \cdot |dz| \\ & = \frac{2Mr^{n-1}}{2\pi} \int_{|z-i\xi|=r} |dz| \\ (2.22) \quad & = 2Mr^n. \end{aligned}$$

Consider the Laurent expansion (2.18). From (2.22) it follows that for all  $n \in \mathbb{Z}$ ,

$$\begin{aligned} \|a_{-(n+1)} + r^{-2} a_{-(n+3)}\| & = \left\| \frac{1}{2\pi i} \int_{|z-i\xi|=r} (z-i\xi)^n f(z) dz + \frac{1}{2\pi i} \int_{|z-i\xi|=r} \frac{(z-i\xi)^{n+2}}{r^2} f(z) dz \right\| \\ & = \left\| \frac{1}{2\pi i} \int_{|z-i\xi|=r} (z-i\xi)^n \left( 1 + \frac{(z-i\xi)^2}{r^2} \right) f(z) dz \right\| \\ (2.23) \quad & \leq 2Mr^n. \end{aligned}$$

Multiplying both sides by  $r^2$  gives (2.20). The lemma is proven.  $\square$

Below we offer a simple proof of an extension of the Gelfand Theorem for groups of isometries (see e.g. [2], [4], [39] for more information about this theorem and applications). For the reduced spectrum of not necessarily uniformly continuous functions that is presented below this extension will play exactly the role of the Gelfand Theorem for groups of isometries in the study of the reduced spectrum of uniformly continuous functions in [2, 4].

**Theorem 2.18.** *Let  $A$  be a closed linear operator on a Banach space  $\mathbb{X}$  such that*

- i)  $\sigma(A) \subset i\mathbb{R}$ ;
- ii) *For some  $\lambda$ -independent positive number  $M$ , the following condition holds*

$$(2.24) \quad \|R(\lambda, A)\| \leq \frac{M}{|\operatorname{Re} \lambda|}, \quad \text{for all } \operatorname{Re} \lambda \neq 0.$$

*Then, the following assertions hold*

- i) *If  $\lambda_0 = i\xi \in i\mathbb{R}$  is an isolated point of  $\sigma(A)$ , then it is an eigenvalue of  $\sigma(A)$ ;*
- ii) *If  $\mathbb{X}$  is non-trivial, then  $\sigma(A) \neq \emptyset$ ;*
- iii) *If  $\sigma(A) = \{0\}$ , then  $A = 0$ .*

*Proof.* i) Set  $f(\lambda) = R(\lambda, A)$ . Since  $i\xi \in i\mathbb{R}$  is an isolated point in  $\sigma(A)$ , it is an isolated singular point of  $f(\lambda)$ , so by Lemma 2.17

$$(2.25) \quad \|r^2 a_{-(n+1)} + a_{-(n+3)}\| \leq 2Mr^{n+2}, \quad n \in \mathbb{Z}.$$

Letting  $r$  tend to 0 in (2.20), we come up with  $a_{-k} = 0$  for all  $k \geq 2$ . This shows that  $i\xi$  is a pole of first order of the resolvent  $f(\lambda) := R(\lambda, A)$ . And hence, by a well known result in Functional Analysis (see e.g. [48, Theorem 5.8 A, p. 306], or, [51, Theorem 3, p. 229]),  $i\xi$  is an eigenvalue of the operator  $A$ . So, the first assertion is proved.

ii) Next, suppose that  $\rho(A) = \mathbb{C}$ . Consider the Laurent expansion (2.18) of  $f(\lambda) = R(\lambda, A)$  at  $\lambda = i\xi$ . Since  $R(\lambda, A)$  is analytic everywhere,  $a_n = 0$  for all  $n \leq -1$ . Note that the formula (2.23) is still valid for this case, and can be re-written in the form

$$(2.26) \quad \|a_{k-1} + r^{-2}a_{k-3}\| \leq 2M\frac{1}{r^k}, \quad k \in \mathbb{Z}.$$

Letting  $r$  tend to infinity we have  $a_n = 0$  whenever  $n \geq 0$ , so  $R(\lambda, A) = 0$ . This is impossible if  $\mathbb{X}$  is non-trivial. This contradiction proves ii).

iii) By (2.20) and (2.26) it is easy to see that all  $a_n = 0$  with  $n \neq -1$ . So,

$$R(\lambda, A) = \frac{a_{-1}}{\lambda}, \quad \text{for all } \lambda \neq 0.$$

We have

$$I = (\lambda - A) R(\lambda, A) = (\lambda - A) \left( \frac{a_{-1}}{\lambda} \right) = a_{-1} - \frac{Aa_{-1}}{\lambda}, \quad \text{for all } \lambda \neq 0.$$

Letting  $\lambda$  tend to infinity we can show that  $a_{-1} = I$ , and thus,  $R(\lambda, A) = I/\lambda$  for  $\lambda \neq 0$ . However, this yields  $I = I - A/\lambda$  for all  $\lambda \neq 0$ , so,  $A = 0$ .  $\square$

**Remark 2.19.** When  $A$  is the generator of a bounded  $C_0$ -group  $(T(t))_{t \in \mathbb{R}}$ , it satisfies the assumptions of these lemmas. In fact, in this case, since

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x dt, \quad x \in \mathbb{X}, \operatorname{Re} \lambda > 0,$$

and

$$R(\lambda, A)x = - \int_0^\infty e^{\lambda t} T(-t)x dt, \quad x \in \mathbb{X}, \operatorname{Re} \lambda < 0,$$

we have

$$\|R(\lambda, A)\| \leq \frac{M}{|\operatorname{Re} \lambda|},$$

where  $M := \sup_{t \in \mathbb{R}} \|T(t)\|$ . Therefore, the above lemmas extend the well-known Gelfand's theorem for  $C_0$ -groups of isometries (see e.g. [4, Corollaries 4.4.8 & 4.4.9]). This will be a key point in the framework for the next results.

**Corollary 2.20.** Let  $\mathcal{F}$  be a subspace of  $BC(\mathbb{R}, \mathbb{X})$  that satisfies Condition F, and let  $f \in BC(\mathbb{R}, \mathbb{X})$ . Then  $sp_{\mathcal{F}}(f) = \emptyset$  if and only if  $f \in \mathcal{F}$ .

*Proof.* If  $f \in \mathcal{F}$ , then the assertion is obvious. Conversely, let  $sp_{\mathcal{F}}(f) = \emptyset$ . Set  $\Lambda = \emptyset$ . If  $\Lambda_{\mathcal{F}}(\mathbb{X})$  is non-trivial, then, by Theorem 2.12,  $\sigma(\tilde{\mathcal{D}}) \subset i\Lambda$ , so  $\sigma(\tilde{\mathcal{D}}) = \emptyset$ . This contradicts Theorem 2.18. Therefore,  $\Lambda_{\mathcal{F}}(\mathbb{X})$  is trivial, and  $f \in \mathcal{F}$ .  $\square$

The following corollary is well known in the spectral theory of functions (see e.g. [43, 4]). However, we will restate it and give a proof based on the our approach to the spectrum.

**Corollary 2.21.** Let  $f \in BC(\mathbb{R}, \mathbb{X})$ . Then  $sp(f) = \{\xi_1, \xi_2, \dots, \xi_N\}$  if and only if  $f$  is of the form  $\mathbb{R} \ni t \mapsto \sum_{k=1}^N a_k e^{i\xi_k t} \in \mathbb{X}$ , where  $0 \neq a_k \in \mathbb{X}$ .

*Proof.* The necessity is obvious. Now we show the sufficiency. Set  $\Lambda = sp(f)$ . By the Riesz decomposition of closed operators (see e.g. [20, Chap. 4]) we can decompose  $\Lambda(\mathbb{X})$  as  $\Lambda(\mathbb{X}) = \Lambda^1 \oplus \dots \oplus \Lambda^N \oplus \Lambda^{N+1}$ , where the spectrum of the restriction of  $\mathcal{D}_\Lambda$  to  $\Lambda^k$ , a closed subspace of  $\Lambda_{\mathcal{F}}(\mathbb{X})$ , is contained in  $\{\xi_k\}$  for all  $k = 1, \dots, N$ , and the spectrum of the restriction of  $\mathcal{D}_\Lambda$  to  $\Lambda^{N+1}$  is empty. Moreover, the restrictions of  $\mathcal{D}_\Lambda$  to  $\Lambda^k$ ,  $k = 1, \dots, N$  are bounded. By Theorem 2.18  $\Lambda^{N+1}$  must be trivial. Therefore, it suffices to show that if  $sp(f) = \{\xi\}$ , then  $f$  is of the form  $ae^{i\xi t}$  with  $a \neq 0$ . Without loss of generality we may assume that  $\xi = 0$ . By Theorem 2.12,  $\sigma(\mathcal{D}_\Lambda) = \{0\}$ , so, by Theorem 2.18,  $\mathcal{D}_\Lambda = 0$ . Since  $\mathcal{D}_\Lambda$  is bounded and  $D(\mathcal{D}) = \Lambda(\mathbb{X})$ , so  $f \in D(\mathcal{D}) = \Lambda(\mathbb{X})$ . And we have  $f' = 0$ . This shows  $f(t) = \text{const}$ . The corollary is proved.  $\square$

**Corollary 2.22.** Let  $f \in BC(\mathbb{R}, \mathbb{X})$ . If  $\xi$  is an isolated point in  $sp(f)$ , then  $\xi \notin sp_{AP(\mathbb{X})}(f)$ .

*Proof.* Set  $\Lambda := sp(f)$ . By Theorem 2.12,  $\Lambda(\mathbb{X})$  can be decomposed as  $\Lambda(\mathbb{X}) = \Lambda_1 \oplus \Lambda_2$ , where the restriction of  $\mathcal{D}_\Lambda$  to  $\Lambda_1$  is bounded and has spectrum as  $\{\xi\}$ , and the restriction of  $\mathcal{D}_\Lambda$  to  $\Lambda_2$  has the spectrum as  $sp(f) \setminus \{\xi\}$ . Therefore,  $f = f_1 + f_2$ , where  $sp(f_1) = \{\xi\}$  and  $sp(f_2) \subset sp(f) \setminus \{\xi\}$ . By Corollary 2.21,  $f_1$  is of the form  $f_1(t) = ae^{i\xi t}$ . Hence,  $\tilde{f} = \tilde{f}_1 = \tilde{f}_2 = \tilde{f}_2$ . By Theorem 2.12  $sp_{AP(\mathbb{X})}(f_2) \subset sp(f_2) \subset sp(f) \setminus \{\xi\}$ . Therefore,  $\xi \notin sp_{AP(\mathbb{X})}(f)$ .  $\square$

The above corollary is known in [5] with additional assumption that  $f \in BUC(\mathbb{R}, \mathbb{X})$ . An immediate consequence of this lemma is the following:

**Corollary 2.23.** *Let  $f \in BC(\mathbb{R}, \mathbb{X})$ . If  $sp(f)$  is discrete, then  $f$  is almost periodic.*

**Remark 2.24.** *This corollary with additional assumption on the uniform continuity of  $f$  has been known in [5, 4], and in a more abstract contexts in [10, 14, 44].*

We are in a position to prove the following that is often referred to as the Loomis Theorem, or, of Loomis Theorem type.

**Theorem 2.25.** *Let  $\mathcal{F}$  be a closed subspace of  $BC(\mathbb{R}, \mathbb{X})$  that satisfies Condition F, and let  $f \in BC(\mathbb{R}, \mathbb{X})$  with countable  $sp_{\mathcal{F}}(f)$ . Then,  $f$  is in  $\mathcal{F}$ .*

*Proof.* Let  $\Lambda := sp_{\mathcal{F}}(f)$ . We will show that  $\Lambda_{\mathcal{F}}(\mathbb{X})$  is trivial. Suppose to the contrary that  $\Lambda_{\mathcal{F}}(\mathbb{X})$  is non-trivial. Then, by Theorem 2.11 and the assumption,  $\sigma(\tilde{\mathcal{D}}_{\Lambda})$  is countable. By Theorem 2.18, it is non-empty, so, since it has an isolated point, it has an eigenvalue, say,  $i\xi$ , where  $\xi \in \mathbb{R}$ . This means that there exists a non-zero  $\tilde{g} \in D(\tilde{\mathcal{D}}_{\Lambda}) \subset \mathbb{Y}$  such that

$$\tilde{\mathcal{D}}\tilde{g} - i\xi\tilde{g} = 0.$$

Therefore, the class  $\tilde{g}$  contains a differentiable function, say  $g \in BC(\mathbb{R}, \mathbb{X})$ , and

$$g' - i\xi g = h \in \mathcal{F}.$$

Using the Variation-of-Constants Formula we have

$$g(t) = e^{i\xi t}g(0) + \int_0^t e^{i\xi(t-\eta)}h(\eta)d\eta, \quad \text{for all } t \in \mathbb{R}.$$

Therefore,

$$e^{-i\xi t}g(t) = g(0) + \int_0^t e^{-i\xi\eta}h(\eta)d\eta, \quad \text{for all } t \in \mathbb{R}.$$

Since the function  $\mathbb{R} \ni \eta \mapsto e^{-i\xi\eta}h(\eta)$  is in  $\mathcal{F}$ , and its primitive  $\mathbb{R} \ni t \mapsto e^{-i\xi t}g(t)$  is bounded, by Condition F, the primitive  $\mathbb{R} \ni t \mapsto e^{-i\xi t}g(t)$  is in  $\mathcal{F}$ . Hence,  $\tilde{g} = \tilde{0}$ . This leads to a contradiction proving that  $\Lambda_{\mathcal{F}}(\mathbb{X})$  is trivial.  $\square$

**Remark 2.26.** *The above theorem has been stated and proved (see [2, 4, 6, 7, 9, 10, 11]) under an additional assumption on the uniform continuity of  $f$ . This assumption is essential for the use of the techniques involving the theory of  $C_0$ -groups.*

Some standard corollaries of Theorem 2.25 are as follows:

**Corollary 2.27.** *Every scalar bounded and continuous function on  $\mathbb{R}$  whose spectrum is countable is almost periodic.*

*Proof.* Let  $\mathcal{F} := AP(\mathbb{R})$ . By the Bohl-Bohr Theorem saying that every bounded primitive of an almost periodic (scalar) function is almost periodic, we can see that  $\mathcal{F}$  satisfies Condition F. Since  $sp_{\mathcal{F}}(u) \subset sp(f)$ ,  $sp_{\mathcal{F}}(f)$  is countable. So, by Theorem 2.25,  $u \in \mathcal{F}$ , that is,  $u$  is almost periodic.  $\square$

**Remark 2.28.** *The above corollary seems to be new even in the scalar case. In fact, in [28] Loomis proved the above corollary in a larger context but with additional assumption on the uniform continuity of  $u$ . The uniform continuity assumption is essential for the techniques used in subsequent extensions (see [2, 4, 6, 7, 9, 10, 11, 45]).*

**Corollary 2.29.** *Let  $\mathbb{X}$  be a Banach space which does not contain any subspace isomorphic to  $c_0$ . Then, every bounded and continuous function with countable spectrum is almost periodic.*

*Proof.* Let  $\mathcal{F} := AP(\mathbb{X})$ . By the Bohl-Bohr-Kadets Theorem, every bounded primitive of an almost periodic function taking values in a Banach space  $\mathbb{X}$  not containing any subspace isomorphic to the space of  $c_0$ , is almost periodic. So, the function space  $\mathcal{F} := AP(\mathbb{X})$  satisfies Condition F. Now by the same argument as in the proof of the above corollary we can prove the corollary.  $\square$

**Remark 2.30.** *The above corollary was first proved by Zhikov (see [25]) with additional assumption on the uniform continuity.*

Let us consider an example with  $\mathcal{F} := AA(\mathbb{X})$ , where  $AA(\mathbb{X})$  denotes the space of all almost automorphic functions introduced by Bochner. For the precise definition and properties of these functions see e.g. [16, 34, 40, 46, 50, 49]. As a special case, the main result in [34] actually says that if  $\mathbb{X}$  does not contain any subspaces isomorphic to  $c_0$ , then each bounded primitive of an  $\mathbb{X}$ -valued almost automorphic function is almost automorphic. That is,  $\mathcal{F} := AA(\mathbb{X})$  satisfies Condition F in this case. Therefore, we arrive at

**Corollary 2.31.** *Let  $\mathbb{X}$  be a Banach space which does not contain any subspaces isomorphic to  $c_0$ , and let  $f$  be in  $BC(\mathbb{R}, \mathbb{X})$  with countable  $sp_{AA(\mathbb{X})}(f)$ . Then,  $f$  is in  $AA(\mathbb{X})$ .*

**Remark 2.32.** *As  $AA(\mathbb{X}) \not\subset BUC(\mathbb{X})$ , the above corollary seems to be new.*

Before closing this subsection we would like to emphasize that we can derive a version of Theorem 2.25 in which  $\mathcal{F}$  satisfies all conditions of Condition F except for the condition iv) that is replaced by an ergodicity condition as discussed in the next section. To avoid repeating the argument in the next subsection we will state the ergodicity condition only for the results for the functions on the half line. The reader can easily adapt them to the entire real line case.

### 3. FUNCTIONS ON THE HALF LINE

Let us consider differential equations of the form

$$(3.1) \quad \dot{x}(t) = \lambda x(t) + f(t), \quad t \in \mathbb{R}^+,$$

where  $f \in BC(\mathbb{R}^+, \mathbb{X})$ . If  $Re\lambda > 0$ , the general solution of (Eq. (3.1)) is

$$(3.2) \quad x(t) = e^{\lambda t}x - \int_t^\infty e^{\lambda(t-s)}f(s)ds, \quad x \in \mathbb{X}, \quad t \in \mathbb{R}^+.$$

Therefore, the only bounded solution of (3.1) is

$$(3.3) \quad x_{\lambda,f}(t) := - \int_t^\infty e^{\lambda(t-s)}f(s)ds, \quad t \in \mathbb{R}^+.$$

On the other hand, if  $Re\lambda < 0$ , the general solution of (3.1) is

$$(3.4) \quad x(t) = e^{\lambda t}x + \int_0^t e^{\lambda(t-s)}f(s)ds, \quad x \in \mathbb{X}, \quad t \in \mathbb{R}^+,$$

so, all solutions in this case are bounded, and all approach zero, except for

$$(3.5) \quad x_{\lambda, f} = \int_0^t e^{\lambda(t-s)} f(s) ds, \quad x \in \mathbb{X}, \quad t \in \mathbb{R}^+.$$

Let us consider a function space  $\mathcal{F} \subset BC(\mathbb{R}^+, \mathbb{X})$  that satisfies the following Condition  $F^+$ :

**Definition 3.1.** *A function space  $\mathcal{F} \subset BC(\mathbb{R}^+, \mathbb{X})$  is said to satisfy Condition  $F^+$  if*

- i) *It is closed, and contains  $C_0(\mathbb{R}^+, \mathbb{X})$ ;*
- ii) *If  $g \in \mathcal{F}$ , then the function  $\mathbb{R}^+ \ni t \mapsto e^{i\xi t} g(t) \in \mathbb{X}$  is in  $\mathcal{F}$  for all  $\xi \in \mathbb{R}$ ;*
- iii) *For each  $h \in \mathcal{F}$ ,  $Re\lambda > 0$ ,  $Re\eta < 0$ , the function  $y(\cdot), z(\cdot)$ , defined as*

$$(3.6) \quad y(t) = \int_t^\infty e^{\lambda(t-s)} h(s) ds, \quad z(t) = \int_0^t e^{\eta(t-s)} h(s) ds, \quad t \in \mathbb{R}^+$$

*are in  $\mathcal{F}$ ;*

- iv) *For each  $B \in L(\mathbb{X})$  and  $f \in \mathcal{F}$ , the function  $Bf(\cdot)$  is in  $\mathcal{F}$ .*

As an example of a function space that satisfies Condition  $F^+$ , we can take  $\mathcal{F} = C_0(\mathbb{R}^+, \mathbb{X})$ . Another function space that satisfies Condition  $F^+$  is  $AA(\mathbb{R}^+, \mathbb{X})$ , the space of all restrictions to  $\mathbb{R}^+$  of the  $\mathbb{X}$ -valued almost automorphic functions. Note that  $AA(\mathbb{R}^+, \mathbb{X})$  contains non-uniformly continuous functions, so it is not a subspace of  $BUC(\mathbb{R}^+, \mathbb{X})$  (see e.g. [34]).

In order to clarify the role condition (iii) of condition  $F^+$  let us consider the differentiation operator  $\mathcal{D}$  on  $BUC(\mathbb{R}^+, \mathbb{X})$  to which we assume  $\mathcal{F}$  belong. It is easy to see that for  $Re\lambda > 0$ ,  $\lambda \in \rho(\mathcal{D})$ , and

$$[R(\lambda, \mathcal{D})f](t) = -x_{\lambda, f}(t) = \int_t^\infty e^{\lambda(t-s)} f(s) ds.$$

Therefore, the first part of condition (iii) means that  $R(\lambda, \mathcal{D})\mathcal{F} \subset \mathcal{F}$  (because  $y = R(\lambda, \mathcal{D})f \in \mathcal{F}$  for each  $f \in \mathcal{F}$ ). Since  $\mathcal{D}$  generates the translation semigroup, using the representation

$$R(\lambda, \mathcal{D})f = \int_0^\infty e^{-\lambda t} S(t) dt, \quad Re\lambda > 0,$$

we can see that this first part of condition (iii) is satisfied if  $\mathcal{F}$  is left invariant under the translation semigroup  $S(t)$  in  $BUC(\mathbb{R}, \mathbb{X})$ , that is,  $S(t)\mathcal{F} \subset \mathcal{F}$  for all  $t \geq 0$ . The inverse is also true from the semigroup theory.

Of course, in  $BUC(\mathbb{R}^+, \mathbb{X})$  the translation  $S(t)$  is not invertible ( $t > 0$ ), and  $\sigma(\mathcal{D})$  contains all complex numbers  $z$  with  $Rez < 0$ . However, the formulas (3.4) and (3.5) give some insights into the structure of solutions of Eq. (3.1). "Within" an asymptotically stable solution  $R(\lambda, \mathcal{D})f$  can be determined uniquely by (3.5) even if  $Re\lambda < 0$ . This leads to the idea of factoring all functions by asymptotically stable functions so that  $Re\lambda < 0$  belongs to the resolvent set of  $\mathcal{D}$  in the quotient space. And this will be all complex plane, but  $i\mathbb{R}$ . Our second part in condition (iii), that says that  $z(t)$  is in  $\mathcal{F}$ , aims at realizing this idea. This is crucial step for us to use the Gelfand Theorem (Theorem 2.18) to study the reduced spectrum of functions on the half line.

In the approach to the reduced spectrum concept via the translation semigroup Arendt and Batty introduced the concept of biinvariance of  $\mathcal{F}$  with respect to the translation semigroup

$(S(t)_{t \geq 0}$  in  $BUC(\mathbb{R}^+, \mathbb{X})$ , that is, the condition  $S(t)\mathcal{F} = \mathcal{F}$  for all  $t \geq 0$  (see e.g. [2, 4]). This yields the surjectiveness of the isometries in the semigroup induced by this translation semigroup. And the Gelfand Theorem can be applied to study the reduced spectrum concept defined in this way.

In the case  $\mathcal{F} \subset BUC(\mathbb{R}^+, \mathbb{X})$  since the induced differentiation operator  $\mathcal{D}$  in the quotient space mentioned above generates the induced translation semigroup we can show that our condition (iii) is equivalent to the biinvariance condition. The advantage of using condition (iii) is clear when it comes to  $BC(\mathbb{R}^+, \mathbb{X})$  in which the translation semigroup is not strongly continuous.

In [8] a concept reduced spectrum of not necessarily uniformly continuous functions on the whole line is also defined. Note that the uniform closedness condition (see [8, Def. 3.1 and Theorem 4.3]) seems to be too restrictive.<sup>3</sup> Moreover, the results need to be adjusted to apply to equations on the half line.

Consider the quotient space  $\mathbb{Y} := BC(\mathbb{R}^+, \mathbb{X})/\mathcal{F}$ . We will use  $\tilde{\mathcal{D}}$  to denote the operator induced by  $\mathcal{D}$  on  $\mathbb{Y}$  which is defined as follows: The domain  $D(\tilde{\mathcal{D}})$  is the set of all classes that contains a differentiable function  $g \in BC(\mathbb{R}^+, \mathbb{X})$  such that  $g' \in BC(\mathbb{R}^+, \mathbb{X})$ ;  $\tilde{\mathcal{D}}\tilde{g} := \tilde{g}'$  for each  $\tilde{g} \in D(\tilde{\mathcal{D}})$ .

By (3.1) and (3.3), and the axiom iii) of Condition  $F^+$ ,

$$(3.7) \quad \begin{aligned} R(\lambda, \tilde{\mathcal{D}})\tilde{f}(t) &= \begin{cases} \int_t^\infty e^{\lambda(t-s)} \tilde{f}(s) ds, & Re\lambda > 0, \ t \in \mathbb{R}^+, \\ -\int_0^t e^{\lambda(t-s)} \tilde{f}(s) ds & Re\lambda < 0, \ t \in \mathbb{R}^+. \end{cases} \\ &= \begin{cases} \int_0^\infty e^{-\lambda\eta} \tilde{f}(t+\eta) d\eta, & Re\lambda > 0, \ t \in \mathbb{R}^+, \\ -\int_0^t e^{\lambda\eta} \tilde{f}(t-\eta) d\eta, & Re\lambda < 0, \ t \in \mathbb{R}^+. \end{cases} \end{aligned}$$

**Lemma 3.2.** *Under the above notations, the operator  $\tilde{\mathcal{D}}$  is a closed operator with  $\sigma(\tilde{\mathcal{D}}) \subset i\mathbb{R}$ . Moreover, for  $Re\lambda \neq 0$ ,*

$$(3.8) \quad \|R(\lambda, \tilde{\mathcal{D}})\| \leq \frac{1}{|Re\lambda|}.$$

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<sup>3</sup>As we have noted above in the theorem cited the authors obviously assume the non-emptiness of the spectrum.

*Proof.* By the above observations, the first assertion of the lemma is obvious. Next, to show (3.8) we can use axiom iii) of Condition  $F^+$ . Therefore, by definition, for  $Re\lambda > 0$ , we have

$$\begin{aligned}
 \|R(\lambda, \tilde{\mathcal{D}})\tilde{f}\|_{\mathbb{Y}} &= \inf_{g \in \mathcal{F}} \sup_{t \in \mathbb{R}^+} \|g(t) + \int_t^\infty e^{\lambda(t-s)} f(s) ds\| \\
 &\leq \inf_{h \in \mathcal{F}} \sup_{t \in \mathbb{R}^+} \left\| \int_t^\infty e^{\lambda(t-s)} [f(s) + h(s)] ds \right\| \\
 &\leq \inf_{h \in \mathcal{F}} \left\| \int_t^\infty e^{Re\lambda(t-s)} ds \|f + h\| \right\| \\
 &= \left\| \int_t^\infty e^{Re\lambda(t-s)} ds \inf_{h \in \mathcal{F}} \|f + h\| \right\| \\
 (3.9) \qquad &= \frac{1}{|Re\lambda|} \|\tilde{f}\|_{\mathbb{Y}}.
 \end{aligned}$$

Similarly, for  $Re\lambda < 0$  we can show that (3.8) holds. This proves the lemma.  $\square$

**Definition 3.3.** Let  $\mathcal{F}$  be a function space that satisfies Condition  $F^+$ , and let  $f \in BC(\mathbb{R}^+, \mathbb{X})$ . Then the reduced spectrum of  $f$  with respect to  $\mathcal{F}$ , denoted by  $sp_{\mathcal{F}}^+(f)$ , is defined to be the set of all reals  $\xi \in \mathbb{R}$  such that  $R(\lambda, \tilde{\mathcal{D}})\tilde{f}$ , as a complex function of  $\lambda$  in  $\mathbb{C} \setminus i\mathbb{R}$ , has no holomorphic extension to any neighborhood of  $i\xi$  in the complex plane.

Since  $sp_{\mathcal{F}}^+(f)$  is the same for all elements  $f$  in a class  $\tilde{g}$ , the use of the notation  $sp_{\mathcal{F}}^+(\tilde{g})$  makes sense.

**Definition 3.4.** Let  $\Lambda$  be a closed subset of the real line. Then,

$$(3.10) \quad \Lambda_{\mathcal{F}}^e(\mathbb{X}) := \{ \tilde{f} \in BC(\mathbb{R}^+, \mathbb{X})/\mathcal{F} : sp_{\mathcal{F}}^+(\tilde{f}) \subset \Lambda, \lim_{\alpha \downarrow 0} \alpha R(\alpha + i\xi, \tilde{\mathcal{D}})\tilde{f} = \tilde{0} \text{ for all } \xi \in \Lambda \}.$$

The property that  $\lim_{\alpha \downarrow 0} \alpha R(\alpha + i\xi, D)f = g \in BC(\mathbb{R}, \mathbb{X})$  is often referred to as the *uniform ergodicity* of  $f$  at  $i\xi \in i\mathbb{R}$ . For related concepts of ergodicity and their equivalence to this one we refer the reader to [3, 4, 6, 18, 45].

Let us consider the restriction  $\tilde{\mathcal{D}}_\Lambda$  of  $\tilde{\mathcal{D}}$  to  $\Lambda_{\mathcal{F}}^e(\mathbb{X})$ .

**Theorem 3.5.** Let  $\Lambda$  be a closed subset of the real line. Then,  $\Lambda_{\mathcal{F}}^e(\mathbb{X})$  is a closed subspace of  $\mathbb{Y} := BC(\mathbb{R}^+, \mathbb{X})/\mathcal{F}$ , and

$$(3.11) \quad \sigma(\tilde{\mathcal{D}}_\Lambda) \subset i\Lambda.$$

*Proof.* To show the closedness of  $\Lambda_{\mathcal{F}}^e(\mathbb{X})$  we assume that  $\{\tilde{f}_n\}_{n=1}^\infty \in \Lambda_{\mathcal{F}}^e(\mathbb{X})$  such that  $\tilde{f}_n \rightarrow \tilde{f} \in \mathbb{Y}$  as  $n \rightarrow \infty$ . Using exactly the argument in the proof of Proposition 2.10 we can easily show that  $sp_{\mathcal{F}}^+(\tilde{f}) \subset \Lambda$ . Next we will show that

$$(3.12) \quad \lim_{\alpha \downarrow 0} \alpha R(\alpha + i\xi, \tilde{\mathcal{D}})\tilde{f} = \tilde{0}.$$



In fact, by the assumption, for each  $\epsilon > 0$  there is a positive integer  $N$  such that if  $n \geq N$ , then,  $\|\tilde{f}_n - \tilde{f}\|_{\mathbb{Y}} < \epsilon$ . So, by (3.7) for each  $\epsilon > 0$  and sufficiently large  $n$ ,

$$\begin{aligned} \limsup_{\alpha \downarrow 0} \alpha \|R(\alpha + i\xi, \tilde{\mathcal{D}})\tilde{f}\|_{\mathbb{Y}} &\leq \limsup_{\alpha \downarrow 0} \alpha \|R(\alpha + i\xi, \tilde{\mathcal{D}})\tilde{f}_n\|_{\mathbb{Y}} + \limsup_{\alpha \downarrow 0} \alpha \|R(\alpha + i\xi, \tilde{\mathcal{D}})(\tilde{f}_n - \tilde{f})\|_{\mathbb{Y}} \\ &\leq 0 + \limsup_{\alpha \downarrow 0} \alpha \int_{\alpha}^{\infty} e^{\alpha t} \|\tilde{f}_n - \tilde{f}\|_{\mathbb{Y}} dt = \|\tilde{f}_n - \tilde{f}\|_{\mathbb{Y}} < \epsilon. \end{aligned}$$

Since  $\epsilon > 0$  is arbitrary, this proves (3.12), yielding the closedness of  $\Lambda_{\mathcal{F}}^e(\mathbb{X})$ .

Now we prove (3.11), by solving  $\tilde{g} \in \Lambda_{\mathcal{F}}^e(\mathbb{X})$  from the equation

$$(3.13) \quad i\beta\tilde{g} - \tilde{\mathcal{D}}_{\Lambda}\tilde{g} = \tilde{f},$$

for each  $\tilde{f} \in \Lambda_{\mathcal{F}}^e(\mathbb{X})$ , and  $\beta \in \mathbb{R}$  such that  $\beta \notin \Lambda$ . First, we show the uniqueness. Assume that  $i\beta\tilde{g} - \tilde{\mathcal{D}}_{\Lambda}\tilde{g} = 0$ . Since  $(i\beta - \tilde{\mathcal{D}})\tilde{g} = 0$ , we have

$$\begin{aligned} (3.14) \quad \tilde{g} &= ((\alpha + i\beta) - \tilde{\mathcal{D}})R(\alpha + i\beta, \tilde{\mathcal{D}})\tilde{g} \\ &= \alpha R(\alpha + i\beta, \tilde{\mathcal{D}})\tilde{g} + (i\beta - \tilde{\mathcal{D}})R(\alpha + i\beta, \tilde{\mathcal{D}})\tilde{g} \\ &= \alpha R(\alpha + i\beta, \tilde{\mathcal{D}})\tilde{g} + R(\alpha + i\beta, \tilde{\mathcal{D}})(i\beta - \tilde{\mathcal{D}})\tilde{g} \\ &= \alpha R(\alpha + i\beta, \tilde{\mathcal{D}})\tilde{g}. \end{aligned}$$

On the other hand, since  $\beta \notin sp_{\mathcal{F}}^+(f)$ , the function  $R(\lambda, \tilde{\mathcal{D}})\tilde{g}$  has an analytic extension to a neighborhood of  $i\beta$ . In particular,  $\lim_{\alpha \downarrow 0} R(\alpha + i\beta, \tilde{\mathcal{D}})\tilde{g}$  exists, so,

$$\tilde{g} = \lim_{\alpha \downarrow 0} \tilde{g} = \lim_{\alpha \downarrow 0} \alpha R(\alpha + i\beta, \tilde{\mathcal{D}})\tilde{g} = 0.$$

Now we prove the existence of a solution to (3.13). Acting as in the proof of Theorem 2.12 we can show that  $\tilde{g} := \lim_{\lambda \rightarrow i\beta} R(\lambda, \tilde{\mathcal{D}})\tilde{f}$  exists as an element of  $\mathbb{Y}$  such that  $i\beta\tilde{g} - \tilde{g}' = \tilde{f}$  and  $sp_{\mathcal{F}}^+(\tilde{g}) \subset sp_{\mathcal{F}}^+(\tilde{f}) \subset \Lambda$ . To complete the proof of the theorem we need to show that  $\lim_{\alpha \downarrow 0} \alpha R(\alpha + i\xi, \tilde{\mathcal{D}})\tilde{g} = \tilde{0}$  for all  $\xi \in \Lambda$ . In fact, for each  $Re\lambda \neq 0$ , we have

$$\begin{aligned} \lim_{\alpha \downarrow 0} \alpha R(\alpha + i\xi, \tilde{\mathcal{D}})R(\lambda, \tilde{\mathcal{D}})\tilde{f} &= \lim_{\alpha \downarrow 0} R(\lambda, \tilde{\mathcal{D}})\alpha R(\alpha + i\xi, \tilde{\mathcal{D}})\tilde{f} \\ &= R(\lambda, \tilde{\mathcal{D}})\lim_{\alpha \downarrow 0} \alpha R(\alpha + i\xi, \tilde{\mathcal{D}})\tilde{f} = 0. \end{aligned}$$

By the above argument used to show (3.12), this shows that  $\lim_{\alpha \downarrow 0} \alpha R(\alpha + i\xi, \tilde{\mathcal{D}})\tilde{g} = 0$ .  $\square$

**Theorem 3.6.** *Let  $\mathcal{F}$  be a function space of  $BC(\mathbb{R}^+, \mathbb{X})$  that satisfies Condition  $F^+$ , and let  $f$  be in  $BC(\mathbb{R}^+, \mathbb{X})$  such that  $sp_{\mathcal{F}}^+(f)$  is countable. Moreover, assume that*

$$(3.15) \quad \lim_{\alpha \downarrow 0} \alpha R(\alpha + i\xi, \tilde{\mathcal{D}})\tilde{f} = 0$$

*for all  $\xi \in sp_{\mathcal{F}}^+(f)$ . Then,  $f \in \mathcal{F}$ .*

*Proof.* Set  $\Lambda := sp_{\mathcal{F}}^+(f)$ . Consider the function space  $\Lambda_{\mathcal{F}}^e(\mathbb{X})$  and the operator  $\tilde{\mathcal{D}}_{\Lambda}$  on it. We are going to prove that the function space  $\Lambda_{\mathcal{F}}^e(\mathbb{X})$  is trivial. In fact, let us assume to the contrary that it is not trivial. Then, since  $sp_{\mathcal{F}}^+(f)$  is countable, by Proposition 3.5, the spectrum  $\sigma(\tilde{\mathcal{D}}_{\Lambda})$  is countable. Therefore, there is an isolated point. By Lemma 3.2 and Theorem 2.18, this isolated point of spectrum of  $\sigma(\tilde{\mathcal{D}}_{\Lambda})$  must be an eigenvalue. And hence, there exists a nonzero vector  $\tilde{g} \in \Lambda_{\mathcal{F}}^e(\mathbb{X})$

such that  $(\tilde{\mathcal{D}} - i\xi)\tilde{g} = 0$ . For  $\alpha > 0$ , since  $(\tilde{\mathcal{D}} - i\xi)\tilde{g} = 0$  by (3.14) we have  $\alpha R(\alpha + i\xi, \tilde{\mathcal{D}})\tilde{g} = \tilde{g}$ . As  $\tilde{g} \in \Lambda_{\mathcal{F}}^e(\mathbb{X})$ ,

$$0 = \lim_{\alpha \rightarrow 0^+} \alpha R(\alpha + i\xi, \tilde{\mathcal{D}})\tilde{g} = \lim_{\alpha \rightarrow 0^+} \tilde{g} = \tilde{g}.$$

This contradiction shows that  $\Lambda_{\mathcal{F}}^e(\mathbb{X})$  must be trivial. Therefore,  $f \in \mathcal{F}$ .  $\square$

**Remark 3.7.** For  $f \in \mathcal{F} \subset BUC(\mathbb{R}^+, \mathbb{X})$ , there is a relation between  $sp_{\mathcal{F}}^+(f)$  and the set  $Sp^+(f)$  of all singularities of the Laplace transform of  $f$ , that is, the set of all reals  $\xi$  such that the Laplace transform  $\hat{f}(\lambda)$  of  $f$  has no analytic extension to any neighborhood of  $i\xi$ . In fact, if  $f \in \mathcal{F} \subset BUC(\mathbb{R}^+, \mathbb{X})$ , then

$$(3.16) \quad sp_{\mathcal{F}}^+(f) \subset Sp^+(f).$$

This is a consequence of [39, Theorem 5.3.4, p. 171]. So, Theorem 3.6 extends [3, Theorem 2.3] and [12, Theorem 4.1].

The following corollaries follow immediately from Theorem 3.6:

**Corollary 3.8.** Let  $f \in BC(\mathbb{R}^+, \mathbb{X})$  such that  $sp_{AAP(\mathbb{X})}^+(f)$  is countable. Moreover, assume that

$$(3.17) \quad \lim_{\alpha \downarrow 0} \alpha R(\alpha + i\xi, \tilde{\mathcal{D}})\tilde{f} = 0$$

for all  $\xi \in sp_{AAP(\mathbb{X})}^+(f)$ . Then,  $f$  is asymptotically almost periodic.

**Corollary 3.9.** Let  $f \in BC(\mathbb{R}^+, \mathbb{X})$  such that  $sp_{AA(\mathbb{X})}^+(f)$  is countable. Moreover, assume that

$$(3.18) \quad \lim_{\alpha \downarrow 0} \alpha R(\alpha + i\xi, \tilde{\mathcal{D}})\tilde{f} = 0$$

for all  $\xi \in sp_{AA(\mathbb{X})}^+(f)$ . Then,  $f$  is asymptotically almost automorphic.

## 4. APPLICATIONS TO THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF EVOLUTION EQUATIONS

**4.1. Equations on the whole line.** Consider evolution equations of the form

$$(4.1) \quad \frac{du(t)}{dt} = Au(t) + f(t), \quad t \in \mathbb{R}, \quad u(t) \in \mathbb{X},$$

where  $A$  is a closed linear operator on a Banach space  $\mathbb{X}$ ,  $f$  is an  $\mathbb{X}$ -bounded and continuous function on  $\mathbb{R}$ . Throughout this section we always assume that  $A$  is a closed linear operator.

**Definition 4.1.** A function  $u \in BC(\mathbb{R}, \mathbb{X})$  is said to be a mild solution of (4.1) if for every  $t \in \mathbb{R}$ ,  $\int_0^t u(s)ds \in D(A)$ , and

$$(4.2) \quad u(t) - u(0) = A \int_0^t u(s)ds + \int_0^t f(s)ds, \quad \text{for all } t \in \mathbb{R}.$$

The following lemma and its proof have been known in the uniform continuity setting (see e.g. [4, 18, 25, 6]). For the reader's convenience we re-state its version for non-uniform continuous mild solutions with a standard proof.

**Lemma 4.2.** *Let  $\mathcal{F} \subset BC(\mathbb{R}, \mathbb{X})$  be a function space that satisfies Condition F, and  $f \in BC(\mathbb{R}, \mathbb{X})$ , and let  $u \in BC(\mathbb{R}, \mathbb{X})$  be a mild solution of (4.1) on  $\mathbb{R}$ . Then,*

$$(4.3) \quad sp_{\mathcal{F}}u \subset \sigma_i(A) \cup sp_{\mathcal{F}}(f),$$

where  $\sigma_i(A) := \{\xi \in \mathbb{R} : i\xi \in \sigma(A)\}$ .

*Proof.* For every  $Re\lambda > 0$ , and  $s \in \mathbb{R}$ , we have

$$\int_0^\infty e^{-\lambda t} \left( \int_0^{t+s} u(\xi) d\xi \right) dt = \frac{1}{\lambda} \left( \int_0^\infty e^{-\lambda t} u(t+s) dt + \int_0^s u(\xi) d\xi \right).$$

Applying this to (4.2), for every  $Re\lambda > 0$ , and  $s \in \mathbb{R}$ , since

$$u(s) = u(0) + \int_0^s u(\xi) d\xi + \int_0^s f(\xi) d\xi, \quad s \in \mathbb{R}$$

we have

$$\begin{aligned} \int_0^\infty e^{-\lambda t} u(t+s) dt &= \int_0^\infty e^{-\lambda t} dt u(0) + A \int_0^\infty e^{-\lambda t} \left( \int_0^{t+s} u(\xi) d\xi \right) dt \\ &\quad + \int_0^\infty e^{-\lambda t} \left( \int_0^{t+s} f(\xi) d\xi \right) dt \\ &= \frac{1}{\lambda} u(0) + \frac{1}{\lambda} A \int_0^\infty e^{-\lambda t} u(t+s) dt + \frac{1}{\lambda} A \int_0^s u(\xi) d\xi \\ &\quad + \frac{1}{\lambda} \int_0^\infty e^{-\lambda t} f(t+s) dt + \frac{1}{\lambda} \int_0^s f(\xi) d\xi \\ &= \frac{1}{\lambda} A \int_0^\infty e^{-\lambda t} u(t+s) dt + \frac{1}{\lambda} \int_0^\infty e^{-\lambda t} f(t+s) dt + \frac{1}{\lambda} u(s). \end{aligned}$$

Therefore, for  $Re\lambda > 0$ , by (2.1),

$$(4.4) \quad R(\lambda, \mathcal{D})u = \frac{1}{\lambda} \mathcal{A} R(\lambda, \mathcal{D})u + \frac{1}{\lambda} R(\lambda, \mathcal{D})f + \frac{1}{\lambda} u,$$

where  $\mathcal{A}$  denotes the operator of multiplication by  $A$  on  $BC(\mathbb{R}, \mathbb{X})$ . Similarly, we can show that (4.4) holds also for  $Re\lambda < 0$ . Therefore, for  $Re\lambda \neq 0$ ,

$$(4.5) \quad (\lambda - \mathcal{A})R(\lambda, \mathcal{D})u = R(\lambda, \mathcal{D})f + u,$$

By the axioms defining Condition F, and the assumption, we arrive at

$$(4.6) \quad (\lambda - \tilde{\mathcal{A}})R(\lambda, \tilde{\mathcal{D}})\tilde{u} = R(\lambda, \tilde{\mathcal{D}})\tilde{f} + \tilde{u}.$$

If  $\xi_0 \in \mathbb{R} \setminus \sigma_i(A)$  and  $\xi_0 \notin sp_{\mathcal{F}}(f)$ , for  $\lambda$  in a small neighborhood of  $i\xi_0$ , and  $Re\lambda \neq 0$ ,

$$(4.7) \quad R(\lambda, \tilde{\mathcal{D}})\tilde{u} = R(\lambda, \tilde{\mathcal{A}})R(\lambda, \tilde{\mathcal{D}})\tilde{f} + R(\lambda, \tilde{\mathcal{A}})\tilde{u}.$$

Therefore,  $R(\lambda, \tilde{\mathcal{D}})\tilde{u}$  has an analytic extension to a neighborhood of  $i\xi_0$ , so  $\xi_0 \notin sp_{\mathcal{F}}(u)$ . This proves the lemma.  $\square$

The following corollary is an immediate consequence of the above lemma and Theorem 2.25.

**Corollary 4.3.** *Let  $\mathcal{F} \subset BC(\mathbb{R}, \mathbb{X})$  be a function space that satisfies Condition F and contains  $f$ , and let  $u \in BC(\mathbb{R}, \mathbb{X})$  be a mild solution of (4.1) on  $\mathbb{R}$ . Moreover, assume that  $\sigma_i(A)$  be countable. Then,  $u$  is in  $\mathcal{F}$ .*

**Remark 4.4.** When  $\mathcal{F} \subset BUC(\mathbb{R}, \mathbb{X})$  and  $u \in BUC(\mathbb{R}, \mathbb{X})$ , the above corollary is known in [2, 6, 10, 11, 25, 45] that extends Loomis Theorem for the scalar functions to vector valued ones. In these works the assumption on the uniform continuity is essential to make use of the techniques based on the spectral properties of  $C_0$ -groups.

The following are standard corollaries of Corollary 4.3.

**Corollary 4.5.** Let  $f \in AP(\mathbb{X})$ ,  $\mathbb{X}$  not contain any subspace isomorphic to  $c_0$ , and let  $u \in BC(\mathbb{R}, \mathbb{X})$  be a mild solution on  $\mathbb{R}$  of (4.1) for which  $\sigma(A) \cap i\mathbb{R}$  is countable. Then,  $u$  is almost periodic.

**Corollary 4.6.** Let  $f \in AA(\mathbb{X})$ ,  $\mathbb{X}$  not contain any subspace isomorphic to  $c_0$ , and let  $u \in BC(\mathbb{R}, \mathbb{X})$  be a mild solution on  $\mathbb{R}$  of (4.1) for which  $\sigma(A) \cap i\mathbb{R}$  is countable. Then,  $u$  is almost automorphic.

**4.2. Equations on the half - line.** In this subsection we consider linear evolution equations on the half line, that is,

$$(4.8) \quad \frac{du(t)}{dt} = Au(t) + f(t), \quad t \in \mathbb{R}^+, \quad u(t) \in \mathbb{X},$$

where  $f \in BC(\mathbb{R}^+, \mathbb{X})$ ,  $A$  is a closed linear operator on  $\mathbb{X}$ .

**Lemma 4.7.** Let  $\mathcal{F} \subset BC(\mathbb{R}^+, \mathbb{X})$  be a function space that satisfies Condition  $F^+$  and contains  $f$ , and let  $u \in BC(\mathbb{R}^+, \mathbb{X})$  be a mild solution of (4.8) on  $\mathbb{R}^+$ . Then,

$$(4.9) \quad sp_{\mathcal{F}}^+ u \subset \sigma_i(A).$$

Moreover, for  $Re\lambda \neq 0$ ,

$$(4.10) \quad R(\lambda, \tilde{\mathcal{D}})\tilde{u} = R(\lambda, \tilde{\mathcal{A}})\tilde{u}.$$

*Proof.* Let  $Re\lambda > 0$ . In the same way as in the proof of Lemma 4.2, by (3.7)

$$(4.11) \quad (\lambda - \tilde{\mathcal{A}})R(\lambda, \tilde{\mathcal{D}})\tilde{u} = R(\lambda, \tilde{\mathcal{D}})\tilde{f} + \tilde{u} = \tilde{u}, \quad Re\lambda > 0.$$

We now show that (4.11) holds for  $Re\lambda < 0$  as well. For each  $Re\lambda < 0$ , using the integration-by-parts formula we have

$$\begin{aligned} \int_0^t e^{\lambda(t-s)} \left( \int_0^s u(\xi) d\xi \right) ds &= -\frac{e^{\lambda(t-s)}}{\lambda} \int_0^s u(\xi) d\xi \Big|_0^t + \frac{1}{\lambda} \int_0^t e^{\lambda(t-s)} u(\xi) d\xi \\ &= -\frac{1}{\lambda} \int_0^t u(\xi) d\xi + \frac{1}{\lambda} \int_0^t e^{\lambda(t-s)} u(\xi) d\xi. \end{aligned}$$

Applying this to (4.2), we arrive at

$$\begin{aligned} \int_0^t e^{\lambda(t-\xi)} u(\xi) d\xi &= \int_0^t e^{\lambda(t-\xi)} u(0) d\xi + A \int_0^t e^{\lambda(t-s)} \left( \int_0^s u(\xi) d\xi \right) ds \\ &\quad + \int_0^t e^{\lambda(t-s)} \left( \int_0^s f(\xi) d\xi \right) ds \\ \int_0^t e^{\lambda(t-\xi)} u(\xi) d\xi &= \frac{e^{\lambda t} u(0)}{\lambda} - \frac{u(0)}{\lambda} + A \left( -\frac{1}{\lambda} \int_0^t u(\xi) d\xi + \frac{1}{\lambda} \int_0^t e^{\lambda(t-s)} u(\xi) d\xi \right) \\ &\quad - \frac{1}{\lambda} \int_0^t f(\xi) d\xi + \frac{1}{\lambda} \int_0^t e^{\lambda(t-s)} f(\xi) d\xi. \end{aligned}$$

Note that for each  $Re\lambda < 0$ ,

$$-\frac{u(0)}{\lambda} - \frac{1}{\lambda} A \int_0^t u(\xi) d\xi - \frac{1}{\lambda} \int_0^t f(\xi) d\xi = -\frac{u(t)}{\lambda}, \quad \text{for all } t \in \mathbb{R}^+.$$

Therefore,

$$\begin{aligned} \int_0^t e^{\lambda(t-\xi)} u(\xi) d\xi &= \frac{1}{\lambda} A \int_0^t e^{\lambda(t-\xi)} u(\xi) d\xi + \frac{1}{\lambda} \int_0^t e^{\lambda(t-\xi)} f(\xi) d\xi \\ &\quad - \frac{1}{\lambda} u(t) + \frac{e^{\lambda t} u(0)}{\lambda}. \end{aligned}$$

Note that

$$\mathbb{R}^+ \ni t \mapsto \frac{1}{\lambda} e^{\lambda t} u(0) \in \mathbb{X} \text{ belongs to } \mathcal{F}.$$

Therefore, by (3.7), for  $Re\lambda < 0$  we have

$$(4.12) \quad -R(\lambda, \tilde{\mathcal{D}})\tilde{u} = -\frac{1}{\lambda} \mathcal{A}R(\lambda, \tilde{\mathcal{D}})\tilde{u} - \frac{1}{\lambda} R(\lambda, \tilde{\mathcal{D}})\tilde{f} - \frac{1}{\lambda} \tilde{u}.$$

So, (4.11) holds for  $Re\lambda < 0$  as well. Next, if  $\xi_0 \in \mathbb{R} \setminus \sigma_i(A)$ , then for  $\lambda$  in a sufficiently small neighborhood of  $i\xi_0$ ,

$$(4.13) \quad R(\lambda, \tilde{\mathcal{D}})\tilde{u} = R(\lambda, \tilde{\mathcal{A}})\tilde{u}.$$

Therefore,  $R(\lambda, \tilde{\mathcal{D}})\tilde{u}$  has an analytic extension to a neighborhood of  $i\xi_0$ , so  $\xi_0 \notin sp_{\mathcal{F}}(u)$ . This proves the lemma.  $\square$

**Remark 4.8.** For  $f, u \in BUC(\mathbb{R}^+, \mathbb{X})$ , the estimate (4.9) has been made in [3] by a different method that seems to be unapplicable to the case of non-uniformly continuous functions  $f$  and  $u$ . In our approach to the spectrum, the resolvent  $R(\lambda, \tilde{\mathcal{D}})\tilde{f}$  for  $Re\lambda < 0$  is explicitly found. So, the above lemma can be proved much easier than in the previous works.

Let  $f \in BC(\mathbb{R}^+, \mathbb{X})$  be uniformly ergodic at  $i\xi$  for some  $\xi \in \mathbb{R}$ , that is, the following limit exists

$$\lim_{\alpha \downarrow 0} \alpha R(\alpha + i\xi, \mathcal{D})f = \bar{f} \in BC(\mathbb{R}^+, \mathbb{X}).$$

Since  $g := R(\alpha + i\xi, \mathcal{D})f$  satisfies the equation  $(\alpha + i\xi)g(t) - g'(t) = f(t)$ , by the Variation-of-Constants Formula,

$$g(t) = e^{(\alpha+i\xi)t} g(0) - \int_0^t e^{(\alpha+i\xi)(t-s)} f(s) ds, \quad \alpha > 0, t \in \mathbb{R}^+.$$

Therefore, for every fixed  $t \in \mathbb{R}^+$ ,

$$\begin{aligned}\bar{f}(t) &= \lim_{\alpha \downarrow 0} \alpha g(t) = \lim_{\alpha \downarrow 0} \alpha e^{(\alpha+i\xi)t} g(0) - \lim_{\alpha \downarrow 0} \alpha \int_0^t e^{(\alpha+i\xi)(t-s)} f(s) ds \\ &= \lim_{\alpha \downarrow 0} \alpha e^{(\alpha+i\xi)t} g(0) \\ &= e^{i\xi t} \bar{f}(0).\end{aligned}$$

This shows that if a function  $f \in BC(\mathbb{R}^+, \mathbb{X})$  is uniformly ergodic at  $i\xi$  for some  $\xi \in \mathbb{R}$ , and

$$\lim_{\alpha \downarrow 0} \alpha R(\alpha + i\xi, \mathcal{D})f = \bar{f} \in BC(\mathbb{R}^+, \mathbb{X}),$$

then for each  $t \in \mathbb{R}^+$ ,  $\bar{f}(t) = e^{i\xi t} a$  for some fixed  $a \in \mathbb{X}$ , so  $\bar{f} \in AAP(\mathbb{R}^+, \mathbb{X})$ .

The following corollaries are obvious.

**Corollary 4.9.** *Let  $\sigma_i(A)$  be countable, and let  $f$  be asymptotically almost periodic. Then every bounded mild solution  $u$  on  $\mathbb{R}^+$  of (4.8) is asymptotically almost periodic provided  $u$  is uniformly ergodic at  $i\xi$  for each  $\xi \in \sigma_i(A)$ .*

**Corollary 4.10.** *Let  $\sigma_i(A)$  be countable, and let  $f$  be asymptotically almost automorphic. Then every bounded mild solution  $u$  of (4.8) on  $[0, \infty)$  is asymptotically almost automorphic provided  $u$  is uniformly ergodic at  $i\xi$  for each  $\xi \in \sigma_i(A)$ .*

Let us consider the homogeneous equation

$$(4.14) \quad \dot{u}(t) = Au(t), \quad u(t) \in \mathbb{X}, \quad t \in \mathbb{R}^+,$$

where  $A$  is a closed linear operator on  $\mathbb{X}$ . A mild solution  $u$  on  $\mathbb{R}^+$  of (4.14) is *asymptotically stable* if  $\lim_{t \rightarrow \infty} u(t) = 0$ .

**Theorem 4.11.** *Let  $u \in BC(\mathbb{R}^+, \mathbb{X})$  be a mild solution of (4.14), and let  $A$  satisfy the following conditions:*

- i)  $\sigma(A) \cap i\mathbb{R}$  is countable;
- ii)  $\lim_{\alpha \downarrow 0} \alpha R(\alpha + i\xi, A)u(t) = 0$  uniformly in  $t \in \mathbb{R}^+$ , for all  $i\xi \in \sigma(A) \cap i\mathbb{R}$ .

*Then, the solution  $u$  is asymptotically stable.*

*Proof.* Let  $u$  be a bounded mild solution on  $\mathbb{R}^+$  of (4.14), and let  $\mathcal{F} := C_0(\mathbb{R}^+, \mathbb{X})$ . Then by Lemma 4.7,  $sp_{C_0(\mathbb{R}^+, \mathbb{X})}^+(f)$  is countable. Moreover, by (4.10), for  $\alpha > 0$ , we have

$$(4.15) \quad \|R(\alpha + i\xi, \tilde{\mathcal{D}})\tilde{u}\| = \|R(\alpha + i\xi, \tilde{\mathcal{A}})\tilde{u}\|.$$

Therefore, for all  $\xi \in \sigma_i(A)$ .

$$(4.16) \quad 0 \leq \lim_{\alpha \downarrow 0} \|\alpha R(\alpha + i\xi, \tilde{\mathcal{D}})\tilde{u}\| = \lim_{\alpha \downarrow 0} \|\alpha R(\alpha + i\xi, \tilde{\mathcal{A}})\tilde{u}\| = 0.$$

Applying Theorem 3.6, we end up with  $u \in C_0(\mathbb{R}^+, \mathbb{X})$ , proving the theorem.  $\square$

The following corollary is an immediate consequence of Theorem 4.11.

**Corollary 4.12.** *Let  $A$  satisfy the following conditions:*

- i)  $\sigma(A) \cap i\mathbb{R}$  is countable;

ii)  $\lim_{\alpha \downarrow 0} \alpha R(\alpha + i\xi, A) = 0$  for all  $i\xi \in \sigma(A) \cap i\mathbb{R}$ .

Then, every bounded mild solution on  $\mathbb{R}^+$  of (4.14) is asymptotically stable.

**Remark 4.13.** If  $\sigma(A) \cap i\mathbb{R} = \emptyset$ , the condition (ii) in the above theorem follows immediately from the condition (i).

If  $A$  is the infinitesimal generator of a bounded  $C_0$ -semigroup, then, as a consequence of Theorem 4.11 we obtain the following well-known Arendt-Batty-Lyubich-Vu Theorem.

**Corollary 4.14.** (The Arendt-Batty-Lyubich-Vu Theorem) *Let  $A$  generate a bounded  $C_0$ -semigroup on a Banach space  $\mathbb{X}$ , and let it satisfy the following conditions:*

- i)  $\sigma(A) \cap i\mathbb{R}$  is countable;
- ii)  $\sigma_p(A^*) \cap i\mathbb{R} = \emptyset$ .

Then, every mild solution on  $\mathbb{R}^+$  of (4.14) is asymptotically stable.

*Proof.* Since  $A$  generates a  $C_0$ -semigroup  $(T(t))_{t \geq 0}$ , each mild solution  $u$  is of the form  $u(t) = T(t)x$ , for all  $t \in \mathbb{R}^+$  and some  $x \in \mathbb{X}$ , so  $u \in BC(\mathbb{R}^+, \mathbb{X})$ . As is well known (see e.g. [4, Sect. 5.5], [13, Proposition 3.2]) the conditions (ii) yields the following

$$\lim_{\alpha \downarrow 0} \alpha R(\alpha + i\xi, A)x = 0, \text{ for all } x \in \mathbb{X}, i\xi \in \sigma(A) \cap i\mathbb{R}.$$

Therefore, for all  $\alpha > 0, i\xi \in \sigma(A) \cap i\mathbb{R}$ ,

$$\begin{aligned} \lim_{\alpha \downarrow 0} \sup_{t \in \mathbb{R}^+} \|\alpha R(\alpha + i\xi, A)u(t)\| &= \lim_{\alpha \downarrow 0} \sup_{t \in \mathbb{R}^+} \|\alpha R(\alpha + i\xi, A)T(t)x\| \\ &= \lim_{\alpha \downarrow 0} \sup_{t \in \mathbb{R}^+} \|\alpha T(t) \int_0^\infty e^{-(\alpha + i\xi)s} T(s)x dt\|, \\ &\leq \lim_{\alpha \downarrow 0} \sup_{t \in \mathbb{R}^+} \|T(t)\| \cdot \alpha \int_0^\infty e^{-(\alpha + i\xi)s} \|T(s)x\| dt \\ &= 0. \end{aligned}$$

So, by Theorem 4.11,  $u$  is asymptotically stable.  $\square$

**Remark 4.15.** The Arendt-Batty-Lyubich-Vu Theorem was proved independently by Arendt and Batty in [1], and Lyubich and Vu in [29]. Earlier in [47] Sklyar<sup>4</sup> and Shirman proved a similar result for bounded  $A$  using a method based on the concept of "isometric limit semigroups" which can be extended to the case where  $A$  is the generator of a  $C_0$ -semigroup. There are many extensions of the Arendt-Batty-Lyubich-Vu Theorem (see e.g. [12, 13, 3, 18]). Note that in all these extensions the assumption on the uniform continuity of mild solutions is essential due to the techniques using the theory of  $C_0$ -semigroups. If  $A$  generates a  $C_0$ -semigroup, the uniform continuity of mild solutions on  $\mathbb{R}^+$  follows from the condition of (ii) in the above corollary.

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